

Quantum Electrodynamics in the Light-Front Weyl Gauge

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January 18, 1996

Abstract

We examine QED(3+1) quantised in the ‘front form’ with finite ‘volume’ regularisation, namely in Discretised Light-Cone Quantisation. Instead of the light-cone or Coulomb gauges, we impose the light-front Weyl gauge $A^- = 0$. The Dirac method is used to arrive at the quantum commutation relations for the independent variables. We apply ‘quantum mechanical gauge fixing’ to implement Gauß’ law, and derive the physical Hamiltonian in terms of unconstrained variables. As in the instant form, this Hamiltonian is invariant under global residual gauge transformations, namely displacements. On the light-cone the symmetry manifests itself quite differently.

PACS number(s): 11.10.Ef, 11.10.Gh, 12.20.Ds

Preprint: MPI-H-V43-1995

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1 Introduction

In recent years there has been something of a resurgence in Hamiltonian field theory particularly with reference to strong interaction physics and gauge theory. This resurgence is partly in order to complement the insight provided by action oriented approaches and the numerical results of lattice gauge theory, partly due to the intuition and experience which the solution to an ‘old-fashioned’ Hamiltonian eigenvalue problem can give, and partly because it has remained somewhat underdeveloped as a formal method within the evolution of quantum field theory. With reference to gauge theories, recent successful applications have been made to QCD in two dimensions [1] and to various theories in higher dimensions [2]. Amongst this (incomplete) list, the additional feature of light-cone quantisation on a null-plane surface of $x^+ \equiv \frac{1}{\sqrt{2}}(x^0 + x^3) = 0$ is quite prominent. The often touted ‘simplicity of the light-cone vacuum’ is one reason for the hope that the fusing of Dirac’s ‘front form’ approach to Hamiltonian dynamics [3] with QCD may lead to sensible results for hadron spectroscopy within a ‘first principles’ calculation.

All the advantages of a light-cone Hamiltonian approach to gauge theory – as a physically insightful method to use – would be lost, of course, in an inappropriate choice of gauge. For various reasons the light-cone gauge $A^+ = 0$ has been the traditional choice in this context. For one, it is one of the class of noncovariant gauges for which Faddeev-Popov ghost decoupling can be argued in an unregulated formalism. It also allows for an operator implementation of the Gauß constraint, so that in some sense the Fock space approach directly gives access to the ‘physical space’. However, the standard arguments turn out to be invalid when one seeks to regularise the theory, especially in the infrared. The method of Discretised Light-Cone Quantisation achieves this by taking space to be of finite volume and imposing respectively periodic and antiperiodic boundary conditions on bosons and fermions. One finds firstly, that the light-cone gauge is no longer accessible due to the ‘gauging away’ of a zero Fourier mode that actually is gauge invariant [4]. The closest gauge choice is then the light-cone Coulomb gauge $\partial_- A^+ = 0$. In addition, one finds that the zero modes satisfy, in general, nontrivial constraint equations [4]. Entangled with the problem of solving these equations is a choice of gauge to remove the rather large freedom untouched by the gauge choice [5, 6]. This problem has been solved, for example in perturbative QED(3+1) in [7] by introducing additional, and complementary, gauge constraints within the light-cone Coulomb gauge. Excluding the purely global zero modes, those that are space independent, the Hamiltonian has been worked out to lowest order in weak coupling.

The difficulty with this approach is that it is not evident how to extend the method to non-Abelian gauge theory, where the problem of finding complementary gauge conditions

to completely eliminate (even topologically trivial) gauge freedom nonperturbatively is expected to be quite difficult. In particular, the task of isolating the ‘fundamental modular domain’ [8] in this framework is the great challenge. So at this juncture, it is fair to ask: is the light-cone Coulomb gauge the most convenient initial gauge choice in order to write down the physical light-cone Hamiltonian in terms of unconstrained variables?

Returning to the ‘instant form’, where quantisation is achieved on a space-like surface at $x^0 = 0$, the Weyl or temporal gauge $A^0 = 0$ is arguably a natural choice. Quantisation is ‘canonical’ and the Hamiltonian is easy to write down, even in QCD. Indeed, it has been available for some time [9]. The outstanding problem here has been the implementation of Gauß’ law on physical states. Some insight has been achieved through works such as [10] or [11]. However, recently a conceptually elegant way has been developed called ‘quantum mechanical gauge fixing’ [12]. Here, Gauß’ law is implemented by a series of unitary transformations on the Hilbert space and operators, including the Gauß law operator, until it becomes essentially trivial to implement the constraint. The result is a non-trivial expression for the Hamiltonian in unconstrained variables (as one would actually hope!) acting in the physical space. This Hamiltonian has now been written down for QCD but remains difficult to solve [13]. Nonetheless, at this stage approximations which do not break local gauge invariance are possible. Indeed, in several simplified settings some physics results have been obtained [14].

The goal of the present work is to derive the physical light-cone Hamiltonian for quantum electrodynamics in the light-front Weyl gauge. That we succeed in this task shows that the light-cone approach can indeed be married to the method of [12] for implementing Gauß’ law. A comment is in order here: already in [12] the method is applied to quantisation on *space-like* surfaces approaching arbitrarily close to the null-plane. In the present work we quantise directly on the null-plane surface. We shall see that the form of the results will be quite different, although we hope to recover essentially the same physics. We return to this point in the discussion. The eventual hope would be that the corresponding light-cone Hamiltonian for QCD, including all zero modes, will be simpler to deal with due to the ‘simple’ vacuum aspect of the light-cone. However, this is still a distant hope given that QED(3+1) will turn out to look more complicated in the present approach.

Consistent with conservation of difficulty, we encounter a problem from the outset: front form Hamiltonian dynamics is a constrained system even in non-gauge theory, and therefore quantisation is already non-canonical. Therefore there is no avoiding something like the Dirac method [15], or analogous methods, in order to obtain consistent (Dirac) brackets for the independent fields thereby enabling the writing down of a quantum theory. It is precisely here that, as in the light-cone Coulomb gauge approach, one must carefully

unravel the various sectors related to normal mode, proper zero modes and global zero modes. Here we are using the decomposition of Fourier space developed in [6]. The resultant formalism becomes complicated for precisely this reason. In particular one must invert rather non-trivial operator equations analogous to those encountered in the instant form [9]. In one case we can give an explicit nonperturbative definition for such an inversion. However, this does not exhaust the constraints one confronts, and in other cases we rely on a formally defined Green's function which, as yet, we have not been able to explicitly give nonperturbatively.

Nonetheless, the Dirac procedure can be performed, leading to disentangling of independent fields, their commutators and the Hamiltonian. The method of [12] is then a somewhat tedious but conceptually straightforward procedure, which we perform here in the 'light-cone gauge representation', here the most natural and possibly only choice, unlike the instant form case [12]. Gauß' law is thus implemented and the Hamiltonian computed.

Despite the abstract form of the representation, some comparisons with other works can be made. Firstly, if the global zero modes are suppressed in our calculation we obtain the same Hamiltonian as in the light-cone Coulomb gauge [6, 7] where the latter work omits these modes of total zero momentum (or spatially constant fields). In fact, we give the complete Hamiltonian including *all* modes. Our work represents the first case where all zero mode sectors in a (3+1) dimensional gauge theory have been comprehensively included in a unified calculation within the framework of light-cone quantisation. Secondly, we are able to demonstrate the existence of a residual displacement symmetry related to large gauge transformations. In instant form QED it has been suggested that the photon can be interpreted as a Goldstone boson associated with spontaneous breaking of precisely this symmetry [12]. Here, matters are more involved due to the quite different manifestation of this symmetry on the light cone.

The paper is organised as follows. In the next section we explain the procedure by which we arrive at a consistent formulation of the quantum theory in the presence of constraints. In section three we use unitary transformations to implement Gauß' law and arrive at the Hamiltonian acting in the physical space. In section four we discuss the displacement symmetry. There is a discussion and a brief statement of conclusions at the end.

2 Quantisation

Notation. In the following, the formalism will have a tendency to become rapidly complicated. We therefore take the pain from the outset to establish a notation that will keep

formulae as simple and as transparent to the physics as much as possible.

Our light-cone convention is that of Kogut and Soper [17]: $x^\pm \equiv \frac{1}{\sqrt{2}}(x^0 \pm x^3)$ with $\partial^\pm = \partial_\mp$. The light-cone *time* variable is x^+ and since we shall work on the surface $x^+ = 0$, it shall be suppressed in the following. The transverse directions (x_1, x_2) are represented by x_\perp . The combined space components (x^-, x_\perp) are represented as \vec{x} . Space is taken to be a ‘hypertorus’ with $-L < x^- < L$ and $-L_\perp < x_\perp < L_\perp$. Bosons will be assigned periodic and fermions antiperiodic boundary conditions. This coupled with light-cone parity means that fermions will have no zero momentum component in any direction.

However, the gauge bosons will have zero modes, and it is these we now disentangle. We follow the structure introduced in [6]. First, we denote the full photon field as $V_\mu(\vec{x})$. This can be expanded in Fourier modes. We distinguish the *simple* zero mode

$$\int_{-L}^{+L} \frac{dx^-}{2L} V_\mu(x^-, x_\perp) , \quad (2.1)$$

via which we build the *normal mode* gauge field

$$A_\mu(\vec{x}) \equiv V_\mu(\vec{x}) - \int_{-L}^{+L} \frac{dx^-}{2L} V_\mu(x^-, x_\perp) . \quad (2.2)$$

These degrees of freedom are known to represent the usual propagating photons in the light-cone representation, and as such we reserve the symbol A_μ to denote them. From the simple zero mode one can build the totally space-independent modes or *global* zero modes

$$q_\mu = \int_{-L}^L \int_{-L_\perp}^{L_\perp} \frac{dx^- d^2 x_\perp}{8LL_\perp^2} V_\mu(\vec{x}) \quad (2.3)$$

where in future we shall write d^3x for $dx^- d^2 x_\perp$ and suppress the limits of integration. Evidently, the q_μ are 0+1 dimensional fields, namely quantum mechanical variables: thus the notation q . Finally, the simple and global zero modes can be used to build modes with *no* x^- -dependence but *no constant* part in x_\perp :

$$a_\mu(x_\perp) = \int_{-L}^{+L} \frac{dx^-}{2L} V_\mu(x^-, x_\perp) - q_\mu . \quad (2.4)$$

These are called the *proper* zero modes. The decomposition is now complete, and in the sequel we shall refer to one of the normal A_μ , proper zero mode a_μ and global zero mode q_μ sectors.

With the above organisation of sectors we must now define the corresponding delta functions. We adopt the notation that the *periodic* three-dimensional delta function is represented as $\delta^{(3)}(\vec{x} - \vec{y})$, which includes the zero modes. For the *antiperiodic* delta function, a subscript ‘a’ is appended: $\delta_a^{(3)}(\vec{x} - \vec{y})$. The explicit difference between these two objects can be easily seen by expanding in discrete Fourier modes. Next we must distinguish the delta functions appropriate for each mode sector for periodic functions.

Thus in the normal mode sector we must subtract the x^- independent part of $\delta^{(3)}$, and so define

$$\delta_n^{(3)}(\vec{x} - \vec{y}) \equiv \delta^{(3)}(\vec{x} - \vec{y}) - \frac{1}{2L} \delta^{(2)}(x_\perp - y_\perp) . \quad (2.5)$$

In the proper zero mode sector, which lives in the two-dimensional perpendicular space, we must subtract the overall two-dimensional volume factor in defining the relevant delta distribution:

$$\delta_p^{(2)}(x_\perp - y_\perp) \equiv \delta^{(2)}(x_\perp - y_\perp) - \frac{1}{4L_\perp^2} . \quad (2.6)$$

One can show that these satisfy completeness in each of their respective sectors.

Now one can safely invert differential operators such as ∂_- and ∂_\perp^2 in terms of well-defined Green's functions, taking care of the respective mode sector in which the operator acts. It will be necessary to be fairly general here. With the covariant derivative $D_\mu = \partial_\mu + ieV_\mu$, we define $\mathcal{G}_{(-)}[\vec{x}, \vec{y}; V_-]$ as the operator-valued Green's function to (iD_-) :

$$(iD_-^x) \mathcal{G}_{(-)}[\vec{x}, \vec{y}; V_-] \equiv \delta^{(3)}(\vec{x} - \vec{y}) - [Sub.] , \quad (2.7)$$

where $[Sub.]$ denotes possible subtractions corresponding to zero eigenvalues of the operator in question. For example, in case of the periodic Green's function with zero functional argument, $[Sub] = \frac{1}{2L} \delta^{(2)}(x_\perp - y_\perp)$. The perpendicular part of the Green's function is actually just $\delta^{(2)}(x_\perp - y_\perp)$ and could be factored out, but keeping it in the present form makes resulting expressions simpler to write down. The operator aspect of $\mathcal{G}_{(-)}$ can be understood, for example, order by order in perturbation theory by building with the Green's function to $(i\partial_-)$, namely $\mathcal{G}_{(-)}[\vec{x}, \vec{y}; 0]$, which only makes sense in the normal mode sector. A nonperturbative construction for this particular Green's function is described in Appendix A. The Green's function $\mathcal{G}_{(\perp)}[x_\perp, y_\perp; \mathcal{O}]$ is defined by the relation

$$[\Delta_\perp^x - \frac{e^2}{2L} \mathcal{O}(x_\perp)] \mathcal{G}_{(\perp)}[x_\perp, y_\perp; \mathcal{O}] \equiv \delta^{(2)}(x_\perp - y_\perp) \quad (2.8)$$

where $\Delta_\perp \equiv \partial_\perp^2$, and now \mathcal{O} is some field operator of mass dimension one. Again, Eq.(2.8) can be elucidated order by order in perturbation theory for which one uses the basic inversion of Δ_\perp via $\mathcal{G}_{(\perp)}[x_\perp, y_\perp; 0]$. A nonperturbative definition is however nontrivial and therefore the above operation is, at best, merely formal and its concrete implementation remains an open problem.

These suffice to cover all the Green's functions we will require in this work. Later it will be useful to have a compact notation for convolutions of the above Green's functions with other quantities

$$(\mathcal{G}_{(\perp)}[\mathcal{O}] * \mathcal{P})(\vec{x}) \equiv \int d^2 y_\perp \mathcal{G}_{(\perp)}[x_\perp, y_\perp; \mathcal{O}] \mathcal{P}(y_\perp, x^-) \quad (2.9)$$

$$(\mathcal{G}_{(-)}[\mathcal{O}] * \mathcal{P})(\vec{x}) \equiv \int d^3 y \mathcal{G}_{(-)}[\vec{x}, \vec{y}; \mathcal{O}] \mathcal{P}(\vec{y}) \quad (2.10)$$

for field operators \mathcal{O} and \mathcal{P} .

We now correspondingly decompose the canonical momenta

$$\Pi_V^\mu(\vec{x}) = \frac{\delta L}{\delta(\partial_+ V_\mu(\vec{x}))} \quad (2.11)$$

into the three sectors of normal, proper zero and global zero modes. We shall give the Lagrangian explicitly shortly. We distinguish the following momenta

$$\Pi^\mu \equiv \frac{\delta L}{\delta(\partial_+ A_\mu)} \quad (2.12)$$

$$\pi^\mu \equiv \frac{\delta L}{\delta(\partial_+ a_\mu)} \quad (2.13)$$

$$p^\mu \equiv \frac{\delta L}{\delta(\partial_+ q_\mu)} . \quad (2.14)$$

Note that the integral expressions for π and p in terms of Π_V^μ do *not* explicitly contain the factors $1/(2L)$, $1/(8LL_\perp^2)$, respectively. Again, the notation in the global zero mode sector emphasises its resemblance to quantum mechanics.

Turning to fermions, ψ , as usual we decompose in spinor space using projectors $\Lambda^\pm \equiv \gamma^0 \gamma^\pm / \sqrt{2}$ leading to bispinors $\psi_\pm \equiv \Lambda^\pm \psi$. Since the fermions are taken to be antiperiodic there is no need to distinguish different Fourier sectors in these.

This completes the introduction to the basic structures needed in the formalism. Further definitions are given, where needed, in the course of calculation or in the Appendices. Reaching a Canonical Formulation. The QED Lagrangian, expressed in terms of the complete fields V_μ , ψ and ψ^\dagger , takes the standard form

$$L = \int d^3x \left[-\frac{1}{4}(\partial_\mu V_\nu - \partial_\nu V_\mu)(\partial^\mu V^\nu - \partial^\nu V^\mu) + \bar{\psi}(i\gamma^\mu \partial_\mu - e\gamma^\mu V_\mu - m)\psi \right] . \quad (2.15)$$

Once the boson field is decomposed into its different sectors

$$V_\mu(\vec{x}) = A_\mu(\vec{x}) + a_\mu(x_\perp) + q_\mu , \quad (2.16)$$

the Lagrangian Eq.(2.15) breaks into three parts

$$L = L_{nm} + L_{pzm} + L_{gzm} , \quad (2.17)$$

where

$$L_{nm} = \int d^3x \left[-\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) + \bar{\psi}(i\gamma^\mu \partial_\mu - e\gamma^\mu A_\mu - m)\psi \right] \quad (2.18)$$

$$L_{pzm} = \int d^3x \left[-\frac{1}{4}(\partial_\mu a_\nu - \partial_\nu a_\mu)(\partial^\mu a^\nu - \partial^\nu a^\mu) - e\bar{\psi}\gamma^\mu \psi a_\mu \right] \quad (2.19)$$

$$L_{gzm} = (8LL_\perp^2) \frac{1}{2}(\partial_+ q_-)^2 - eq_\mu \int d^3x \bar{\psi}\gamma^\mu \psi . \quad (2.20)$$

These formulae in turn can be simplified by decomposing the total electromagnetic current, $\mathcal{J}^\mu = -e\bar{\psi}\gamma^\mu\psi$, into its normal mode and proper and global zero mode parts:

$$\mathcal{J}^\mu = J^\mu + j^\mu + Q^\mu . \quad (2.21)$$

This is given in detail in Appendix B.

Now we may outline our method of quantisation. The Weyl gauge condition will be imposed strongly, namely we set $A_+ = a_+ = q_+ = 0$ at the classical level. As usual, this means Gauß' law (given explicitly later) appears as a constraint to be imposed on states. However, mixed in with this constraint will be many other constraints peculiar to the light-cone approach. In this section we will eliminate these (second class) constraints while leaving the (first class) Gauß constraint untouched. Our procedure for doing this is as follows: We will analyse different subsystems, where only one field sector is treated in terms of independent degrees of freedom while the remaining fields are regarded as non-dynamical external fields and/or currents. Then we will exchange non-dynamical modes for effective interactions of dynamical ones and will give a simpler (though non-local) Lagrangian where only dynamical fields are present. This will result in a sequence of equivalent effective Lagrangians which contain fewer modes but have Euler-Lagrange equations the same as those which are generated by the primary Lagrangian provided the constraint equations are implemented for nondynamical fields. This procedure is based on the observation that different Lagrangians can lead to the same system of Euler-Lagrange equations though they may have very different constraint structure (in a different context see, for example, [18]). One can feel free to choose the most suitable one for carrying out the canonical quantization procedure. In more detail, we shall start with the proper zero mode Lagrangian Eq.(2.19) and analyse its canonical structure. When the Dirac brackets are found then another equivalent Lagrangian will be proposed which contains only dynamical modes (these having non-zero brackets) and effective non-local terms. A similar analysis can be done for the Lagrangian Eq.(2.18) and another effective Lagrangian for dynamical modes can be found. Then the sector of global zero modes and fermions are analysed leading to the final effective Lagrangian. Gathering these partial effective Lagrangians we obtain a total effective Lagrangian which contains only the Gauß constraint. However, the brackets/commutators will mostly take the noncanonical form usually seen in the light-front. This is a consequence of the implementation in this method of the typically light-front second class constraints for the original Lagrangian.

Step 1: Proper zero mode sector. The Lagrangian Eq.(2.19) can be written explicitly in light-front coordinates

$$L_{Weyl}^{pzm} = \int d^3x \left[-\partial_i a_- \partial_+ a_i + \frac{1}{2}(\partial_+ a_-)^2 - \frac{1}{4}(\partial_i a_j - \partial_j a_i)^2 + j^\perp a_\perp + j^- a_- \right] , \quad (2.22)$$

where the Weyl gauge condition $a_+ = 0$ has been explicitly imposed and the proper zero modes of currents are given explicitly in Appendix B. This Lagrangian leads to the classical equations of motion

$$\partial_+^2 a_- = \partial_i \partial_+ a_i + j^- \quad (2.23)$$

$$-\partial_i \partial_+ a_- = \Delta_\perp a_i - \partial_i \partial_k a_k + j^i \quad (2.24)$$

and the canonical conjugate momenta

$$\pi^-(x_\perp) = \frac{\delta L_{Weyl}^{pzm}}{\delta(\partial_+ a_-(x_\perp))} = 2L \partial_+ a_-(x_\perp) \quad (2.25)$$

$$\pi^i(x_\perp) = \frac{\delta L_{Weyl}^{pzm}}{\delta(\partial_+ a_i(x_\perp))} = -2L \partial_i a_-(x_\perp) \quad (2.26)$$

for the independent gauge fields. Though our gauge choice has eliminated one primary constraint connected with the gauge potential a_+ , there is still another constraint Eq.(2.26) and we must implement the Dirac constraint procedure here. The constraint Eq.(2.26) appears in the extended Hamiltonian density via Lagrange multiplier fields $u_i(x_\perp)$

$$\begin{aligned} \mathcal{H}_E^{pzm} &= \frac{1}{2L} \partial_+ a_- \pi^- + \frac{1}{2L} \partial_+ a_i \pi^i - \mathcal{L}_{Weyl}^{pzm} + u_i(\pi^i + 2L \partial_i a_-) \\ &= \frac{1}{2} \left(\frac{\pi^-}{2L} \right)^2 + \frac{1}{4} (\partial_i a_j - \partial_j a_i)^2 - a_- j^- - a_i j^i + u_i(\pi^i + 2L \partial_i a_-) \end{aligned} \quad (2.27)$$

and there is a sequence of constraints

$$\phi_1^i = \pi^i + 2L \partial_i a_- \simeq 0 \quad (2.28)$$

$$\phi_2^i = \Delta_\perp a_i - \partial_i \partial_j a_j + j^i + \frac{1}{2L} \partial_i \pi^- \simeq 0, \quad (2.29)$$

which allow one to eliminate dependent degrees of freedom. While Eq.(2.28) can be easily solved for π^i , Eq.(2.29) is more involved. Its transverse projection forms a constraint on the transverse part of a_i which can be trivially solved to give:

$$a_i^T = \mathcal{G}_{(\perp)}[0] * j_i^T. \quad (2.30)$$

Here, the superscript T indicates the transverse projection. The longitudinal part of a_i is *not* constrained by Eq.(2.29). This suggests it is part of a dynamical field. It is convenient to define a new field, $\pi(x_\perp)$, in terms of a_i^L , the longitudinal projection of a_i , via a decomposition

$$a_i^L = \mathcal{G}_{(\perp)}[0] * \left(j_i^L - \frac{1}{2L} \partial_i \pi \right). \quad (2.31)$$

This equation defines the field π . One can thus reorganise the decomposition $a_i = a_i^T + a_i^L$ by substituting Eqs.(2.30,2.31), and obtain (writing the expression explicitly)

$$a_i(x_\perp) = - \int d^2 y_\perp \mathcal{G}_{(\perp)}[x_\perp, y_\perp; 0] \left(j^i(y_\perp) + \frac{1}{2L} \partial_i \pi(y_\perp) \right). \quad (2.32)$$

It will turn out later on that the field π has no explicit dependence on j^i and is a good candidate for the canonical variable. Next, the longitudinal projection of the constraint Eq.(2.29) can be solved for π^-

$$\pi^-(x_\perp) = 2L \int d^2 y_\perp \mathcal{G}_{(\perp)}[x_\perp, y_\perp; 0] \partial_j j^j(y_\perp). \quad (2.33)$$

The Dirac brackets for independent fields are nonzero only for the pair (π, a_-)

$$\{\pi(x_\perp), a_-(y_\perp)\}_D = -\delta_p^{(2)}(x_\perp - y_\perp) \quad (2.34)$$

and the Dirac Hamiltonian at the end of step one contains the effective nonlocal terms

$$\begin{aligned} H_D^{pzm} &= \frac{2L}{2} \int d^2 x_\perp \int d^2 y_\perp j^i(x_\perp) \mathcal{G}_{(\perp)}[x_\perp, y_\perp; 0] j^i(y_\perp) \\ &- \int d^2 x_\perp \int d^2 y_\perp \pi(x_\perp) \mathcal{G}_{(\perp)}[x_\perp, y_\perp; 0] \partial_i j^i(y_\perp) - 2L \int d^2 x_\perp a_-(x_\perp) j^-(x_\perp). \end{aligned} \quad (2.35)$$

One can check that the effective equations of motion which follow from the above Dirac Hamiltonian and brackets agree with the Euler-Lagrange equations Eqs.(2.23-2.24). Therefore our Dirac Hamiltonian and brackets describe the same classical system as the primary Lagrangian Eq.(2.22) and one can give an equivalent Lagrangian density

$$\mathcal{L}_{eff}^{pzm} = \frac{1}{2L} \pi \partial_+ a_- - \mathcal{H}_D^{pzm} = \frac{1}{2L} \pi \partial_+ a_- + \frac{1}{2L} \pi \mathcal{G}_{(\perp)}[0] * \partial_i j^i + a_- j^- - \frac{1}{2} j^i \mathcal{G}_{(\perp)}[0] * j^i, \quad (2.36)$$

which straightforwardly leads to the correct Dirac brackets and equations of motion.

Step 2: Normal mode sector. In the second step, let us analyse the sector of normal modes of gauge field potentials A_μ treating normal modes of electromagnetic currents J^μ as arbitrary external sources. From the Lagrangian Eq.(2.18) we take these terms which contain A_μ

$$\mathcal{L}_{Weyl}^{nm} = \partial_+ A_i (\partial_- A_i - \partial_i A_-) + \frac{1}{2} (\partial_+ A_-)^2 - \frac{1}{4} (\partial_i A_j - \partial_j A_i)^2 + A_- J^- + A_i J^i, \quad (2.37)$$

and find the Euler-Lagrange equations of motion

$$\partial_+ (\partial_+ A_- - \partial_i A_i) = J^- \quad (2.38)$$

$$(2\partial_+ \partial_- - \Delta_\perp) A_i = \partial_i (\partial_+ A_- - \partial_j A_j) + J^i \quad (2.39)$$

and the canonical momenta

$$\Pi^- = \frac{\delta L_{Weyl}^{nm}}{\delta(\partial_+ A_-)} = \partial_+ A_- \quad (2.40)$$

$$\Pi^i = \frac{\delta L_{Weyl}^{nm}}{\delta(\partial_+ A_i)} = \partial_- A_i - \partial_i A_- . \quad (2.41)$$

Here there is only one constraint

$$\Phi^i = \Pi^i - \partial_- A_i + \partial_i A_- \simeq 0 \quad (2.42)$$

which is second class, and the unconstrained variables have non-trivial Dirac brackets

$$\{A_-(\vec{x}), \Pi^-(\vec{y})\}_D = \delta_n^{(3)}(\vec{x} - \vec{y}) \quad (2.43)$$

$$\{A_i(\vec{x}), \Pi^-(\vec{y})\}_D = \frac{i}{2} \partial_i^x \mathcal{G}_{(-)}[\vec{x}, \vec{y}; 0] \quad (2.44)$$

$$\{\Pi^-(\vec{x}), \Pi^-(\vec{y})\}_D = \frac{i}{2} \Delta_\perp \mathcal{G}_{(-)}[\vec{x}, \vec{y}; 0] \quad (2.45)$$

$$\{A_i(\vec{x}), A_j(\vec{y})\}_D = -\frac{i}{2} \delta_{ij} \mathcal{G}_{(-)}[\vec{x}, \vec{y}; 0] . \quad (2.46)$$

The Dirac Hamiltonian

$$\mathcal{H}_D^{nm'} = \frac{1}{2}(\Pi^-)^2 + \frac{1}{4}(\partial_i A_j - \partial_j A_i)^2 - A_- J^- - A_i J^i, \quad (2.47)$$

generates Hamilton's equations of motion for unconstrained variables which agree with the Lagrange equations Eqs.(2.38-2.39). One may define a new field $\Pi = \Pi^- - \partial_i A_i$, which simplifies the structure of Dirac brackets

$$\{A_-(\vec{x}), \Pi(\vec{y})\}_D = \delta_n^{(3)}(\vec{x} - \vec{y}) \quad (2.48)$$

$$\{A_i(\vec{x}), A_j(\vec{y})\}_D = -\frac{i}{2} \delta_{ij} \mathcal{G}_{(-)}[\vec{x}, \vec{y}; 0] , \quad (2.49)$$

and then all other brackets vanish. This new Dirac Hamiltonian

$$\mathcal{H}_D^{nm} = \frac{1}{2}(\Pi)^2 + \Pi \partial_i A_i + \frac{1}{2}(\partial_i A_j)^2 - A_- J^- - A_i J^i \quad (2.50)$$

generates equations of motion which are equivalent to the previous ones and can be used for defining an effective Lagrangian

$$\begin{aligned} \mathcal{L}_{eff}^{nm} &= \partial_+ A_i \partial_- A_i + \Pi \partial_+ A_- - \mathcal{H}_D \\ &= \partial_+ A_i \partial_- A_i - \frac{1}{2}(\partial_i A_j)^2 - \frac{1}{2}(\Pi)^2 + \Pi(\partial_+ A_- - \partial_i A_i) + A_- J^- + A_i J^i. \end{aligned} \quad (2.51)$$

Having analysed the canonical structure of the gauge field sector one can substitute the Lagrangian Eq.(2.19) by Eq.(2.36) and the boson part of Eq.(2.18) by Eq.(2.51) and instead of the total Lagrangian Eq.(2.17) one can work with the effective Lagrangian

$$\begin{aligned} \mathcal{L}'_{eff} &= \partial_+ A_i \partial_- A_i - \frac{1}{2}(\partial_i A_j)^2 - \frac{1}{2}(\Pi)^2 + \Pi(\partial_+ A_- - \partial_i A_i) + \frac{1}{2L} \pi \partial_+ a_- \\ &+ \frac{1}{2}(\partial_+ q_-)^2 + i\sqrt{2} \psi_+^\dagger \partial_+ \psi_+ + i\sqrt{2} \psi_-^\dagger \partial_- \psi_- + i\psi_-^\dagger \alpha^i \partial_i \psi_+ + i\psi_+^\dagger \alpha^i \partial_i \psi_- \\ &- m\psi_+^\dagger \gamma^0 \psi_- - m\psi_-^\dagger \gamma^0 \psi_+ - e\sqrt{2} \psi_-^\dagger \psi_- V_- \\ &- eV'_i \left(\psi_+^\dagger \alpha^i \psi_- + \psi_-^\dagger \alpha^i \psi_+ \right) - \frac{1}{2} j^i \left(\mathcal{G}_{(\perp)}[0] * j^i \right) . \end{aligned} \quad (2.52)$$

In this expression,

$$V'_i = A_i - \frac{1}{2L} \partial_i \left(\mathcal{G}_{(\perp)}[0] * \pi \right) + q_i, \quad (2.53)$$

namely it is the original V_i but with its proper zero mode a_i expressed as in Eq.(2.32) and the new current-current interaction subtracted. This piece has been explicitly reintroduced as the last term in Eq.(2.52). The decomposition of V_- remains unchanged.

Step 3: Fermion sector. In the next step, we take the fermion part of the effective Lagrangian Eq.(2.52)

$$\begin{aligned}\mathcal{L}_{Weyl}^{fer} &= i\sqrt{2}\psi_+^\dagger\partial_+\psi_+ + i\sqrt{2}\psi_-^\dagger\partial_-\psi_- + i\psi_-^\dagger\alpha^i\partial_i\psi_+ + i\psi_+^\dagger\alpha^i\partial_i\psi_- \\ &- m\psi_+^\dagger\gamma^0\psi_- - m\psi_-^\dagger\gamma^0\psi_+ - e\sqrt{2}\psi_-^\dagger\psi_-V_- \\ &- eV'_i\left(\psi_+^\dagger\alpha^i\psi_- + \psi_-^\dagger\alpha^i\psi_+\right) - \frac{1}{2}j^i\left(\mathcal{G}_{(\perp)}[0]*j^i\right) .\end{aligned}\quad (2.54)$$

Now the system contains many more constraints and these are given in Appendix C. The nondynamical modes are the fermion field ψ_- and the global zero mode q_i . These are determined by the following differential equations and global integral condition

$$\begin{aligned}(i\sqrt{2}\partial_- - e\sqrt{2}V_-)\psi_- &= -i\alpha^i\partial_i\psi_+ + m\gamma^0\psi_+ \\ &+ e\alpha^i\left(V_i - \mathcal{G}_{(\perp)}[0]*j^i\right)\psi_+ ,\end{aligned}\quad (2.55)$$

$$0 = Q^i = \frac{1}{8LL_\perp^2} \int d^3x (\psi_+^\dagger\alpha^i\psi_- + \psi_-^\dagger\alpha^i\psi_+)(\vec{x}). \quad (2.56)$$

First one gets

$$\begin{aligned}\psi_-(\vec{x}) &= \frac{1}{\sqrt{2}}\left(\mathcal{G}_{(-)}[V_-]*\xi\right)(\vec{x}) \\ &- \frac{e}{\sqrt{2}}\left(\mathcal{G}_{(\perp)}[0]*j^i(x_\perp) - q_i\right)\alpha^i\left(\mathcal{G}_{(-)}[V_-]*\psi_+\right)(\vec{x}) ,\end{aligned}\quad (2.57)$$

where

$$\xi(\vec{x}) = \left[m\gamma^0 - i\alpha^i\partial_i + e\alpha^i\left(A_i(\vec{x}) - \frac{1}{2L}\partial_i\left(\mathcal{G}_{(\perp)}[0]*\pi\right)(\vec{x})\right)\right]\psi_+(\vec{x}) . \quad (2.58)$$

Note that these are not yet solutions for the dependent fermion field ψ_- because these fields appear also on the right hand side in the zero mode currents j^i . However one can introduce them into the definition of $j^i + Q^i$ (see Eqs.(B.4,B.7))

$$j^i(x_\perp) + Q^i = -\frac{e}{2L}\Gamma^i(x_\perp) + \frac{e^2}{2L}\mathcal{M}^{ik}(x_\perp)\left((\mathcal{G}_{(\perp)}[0]*j^k)(x_\perp) - q_k\right) , \quad (2.59)$$

where

$$\begin{aligned}\Gamma^i(x_\perp) &= \frac{1}{\sqrt{2}}\int dx^- \psi_+^\dagger(\vec{x})\alpha^i\left(\mathcal{G}_{(-)}[V_-]*\xi\right)(\vec{x}) \\ &+ \frac{1}{\sqrt{2}}\int dx^- \xi^\dagger(\vec{x})\alpha^i\left(\mathcal{G}_{(-)}[V_-]*\psi_+\right)(\vec{x})\end{aligned}\quad (2.60)$$

$$\begin{aligned}\mathcal{M}^{ij}(x_\perp) &= \frac{1}{\sqrt{2}}\int dx^- \psi_+^\dagger(\vec{x})\alpha^i\alpha^j\left(\mathcal{G}_{(-)}[V_-]*\psi_+\right)(\vec{x}) \\ &+ \frac{1}{\sqrt{2}}\int dx^- \psi_+^\dagger(\vec{x})\alpha^j\alpha^i\left(\mathcal{G}_{(-)}[V_-]*\psi_+\right)(\vec{x}) \\ &= \delta^{ij}\sqrt{2}\int dx^- \psi_+^\dagger(\vec{x})\left(\mathcal{G}_{(-)}[V_-]*\psi_+\right)(\vec{x}) = \delta^{ij}\mathcal{M}^2(x_\perp) .\end{aligned}\quad (2.61)$$

Now, from the constraint $Q^i = 0$ one gets the differential equation

$$\left[\Delta_{\perp} - \frac{e^2}{2L} \mathcal{M}^2(x_{\perp}) \right] \left((\mathcal{G}_{(\perp)}[0] * j^i)(x_{\perp}) - q_i \right) = -\frac{e}{2L} \Gamma^i(x_{\perp}), \quad (2.62)$$

which has a formal solution

$$(\mathcal{G}_{(\perp)}[0] * j^i)(x_{\perp}) - q_i = -\frac{e}{2L} \int d^2 y_{\perp} \mathcal{G}_{(\perp)}[x_{\perp}, y_{\perp}; \mathcal{M}^2] \Gamma^i(y_{\perp}) \quad (2.63)$$

in terms of the functional Green's function introduced in Eq.(2.8). Finally, we may express the nondynamical fermion field as

$$\psi_{-}(\vec{x}) = \frac{1}{\sqrt{2}} \left(\mathcal{G}_{(-)}[V_{-}] * \xi \right) (\vec{x}) + \frac{e^2}{2\sqrt{2}L} \left(\mathcal{G}_{(\perp)}[\mathcal{M}^2] * \Gamma^i \right) (x_{\perp}) \left(\mathcal{G}_{(-)}[V_{-}] * \psi_{+} \right) (\vec{x}), \quad (2.64)$$

whereby we obtain

$$H_D^{fer} = \frac{1}{\sqrt{2}} \int d^3 x \xi^{\dagger}(\vec{x}) \left(\mathcal{G}_{(-)}[V_{-}] * \xi \right) (\vec{x}) + \frac{1}{2} \frac{e^2}{2L} \int d^2 x_{\perp} \Gamma^i(x_{\perp}) \left(\mathcal{G}_{(\perp)}[\mathcal{M}^2] * \Gamma^i \right) (x_{\perp}) \quad (2.65)$$

as the Dirac Hamiltonian for unconstrained fields. Just as before we can give the effective Lagrangian for the fermions

$$\begin{aligned} \mathcal{L}_{eff}^{fer} &= (\partial_{+} \psi_{+}) \Pi_{\psi_{+}} - \mathcal{H}_D^{fer} = i\sqrt{2} \psi_{+}^{\dagger} \partial_{+} \psi_{+} \\ &- \frac{1}{\sqrt{2}} \xi^{\dagger} (\mathcal{G}_{(-)}[V_{-}] * \xi) - \frac{1}{2} \frac{e^2}{4L^2} \Gamma^i \left(\mathcal{G}_{(\perp)}[\mathcal{M}^2] * \Gamma^i \right). \end{aligned} \quad (2.66)$$

Quantum theory. Having eliminated all nondynamical fields one may now proceed by substituting the fermion part of Eq.(2.52) by that given in Eq.(2.66). One thereby obtains the total effective Lagrangian

$$\begin{aligned} \mathcal{L}_{eff} &= \partial_{+} A_i \partial_{-} A_i - \frac{1}{2} (\partial_i A_j)^2 - \frac{1}{2} (\Pi)^2 + \Pi (\partial_{+} A_{-} - \partial_i A_i) - \frac{1}{2L} \pi \partial_{+} a_{-} + \frac{1}{2} (\partial_{+} q_{-})^2 \\ &+ i\sqrt{2} \psi_{+}^{\dagger} \partial_i \psi_{+} - \frac{1}{\sqrt{2}} \xi^{\dagger} (\mathcal{G}_{(-)}[V_{-}] * \xi) - \frac{1}{2} \frac{e^2}{4L^2} \Gamma^i \left(\mathcal{G}_{(\perp)}[\mathcal{M}^2] * \Gamma^i \right) \end{aligned} \quad (2.67)$$

as the starting point for the canonical quantisation procedure. It generates Euler-Lagrange equations of motion which agree with the dynamical equations of the primary Lagrangian Eq.(2.15) with all nondynamical equations (formally) implemented. The canonical momenta and Dirac brackets are the same as in the former partial analyses and are given in the Appendix C. They lead to the equal- x^+ quantum commutation relations

$$[\Pi(\vec{x}), A_{-}(\vec{y})] = -i\delta_n^{(3)}(\vec{x} - \vec{y}) \quad (2.68)$$

$$\left\{ \psi_{+}^{\dagger}(\vec{x}), \psi_{+}(\vec{y}) \right\} = \frac{1}{\sqrt{2}} \Lambda^{(+)} \delta_a^{(3)}(\vec{x} - \vec{y}) \quad (2.69)$$

$$[\pi(x_{\perp}), a_{-}(y_{\perp})] = -i\delta_p^{(2)}(x_{\perp} - y_{\perp}) \quad (2.70)$$

$$[A_i(\vec{x}), A_j(\vec{y})] = \frac{\delta_{ij}}{2} \mathcal{G}_{(-)}[\vec{x}, \vec{y}; 0] \quad (2.71)$$

$$[p^-, q_-] = -i \quad (2.72)$$

and the quantum Hamiltonian, like \mathcal{L}_{eff} , which comes from \mathcal{H}_D^{pzm} , \mathcal{H}_D^{nm} and H_D^{fer} ,

$$\begin{aligned} H_{eff} &= \int d^3x \left[\frac{1}{2} (\Pi(\vec{x}))^2 + \Pi \partial_i A_i(\vec{x}) + \frac{1}{2} (\partial_i A_j(\vec{x}))^2 \right] \\ &+ \frac{1}{16LL_\perp^2} (p^-)^2 + \frac{1}{\sqrt{2}} \int d^3x \xi^\dagger(\vec{x}) \left(\mathcal{G}_{(-)} [V_-] * \xi \right) (\vec{x}) \\ &+ \frac{1}{2} \frac{e^2}{2L} \int d^2x_\perp \Gamma^i(x_\perp) \left(\mathcal{G}_{(\perp)} [\mathcal{M}^2] * \Gamma^i \right) (x_\perp), \end{aligned} \quad (2.73)$$

where the formal expressions must be defined by some careful ordering procedure. Due to the effective equations of motion the Gauß' law operator

$$G(\vec{x}) = \partial_- \Pi(\vec{x}) + 2\partial_- \partial_i A_i(\vec{x}) - \Delta_\perp A_-(\vec{x}) - \Delta_\perp a_-(x_\perp) - e\sqrt{2}\psi_+^\dagger(\vec{x})\psi_+(\vec{x}) \quad (2.74)$$

is x^+ -independent. It leads to a classically first class constraint, $G \simeq 0$; namely it must annihilate physical states in the quantum theory. Furthermore, it is intimately connected to the residual gauge symmetry (see Appendix D).

Translation Generators. With the effective Lagrangian density Eq.(2.67) one can calculate the canonical momentum-energy tensor

$$\begin{aligned} T^{\nu\mu} &= \frac{\delta L_{eff}}{\delta (\partial_\nu A_i)} \partial^\mu A_i + \partial^\mu \psi_+ \frac{\delta L_{eff}}{\delta (\partial_\nu \psi_+)} + \frac{1}{2L} \frac{\delta L_{eff}}{\delta (\partial_\nu \pi)} \partial^\mu \pi \\ &+ \frac{1}{2L} \frac{\delta L_{eff}}{\delta (\partial_\nu a_-)} \partial^\mu a_- + \frac{1}{8LL_\perp^2} \frac{\delta L_{eff}}{\delta (\partial_\nu q_-)} \partial^\mu q_- - g^{\nu\mu} \mathcal{L}_{eff}. \end{aligned} \quad (2.75)$$

Then from the generators of translations $P^\mu = \int d^3x T^{+\mu}(\vec{x})$, the spatial translations are

$$P^i = \int d^3x \left[-\partial_- A_k \partial_i A_k - \Pi \partial_i A_- - i\sqrt{2}\psi_+^\dagger \partial_i \psi_+ - \frac{1}{2L} \pi \partial_i a_- \right] \quad (2.76)$$

$$P^+ = \int d^3x \left[\partial_- A_k \partial_- A_k + \Pi \partial_- A_- + i\sqrt{2}\psi_+^\dagger \partial_- \psi_+ \right]. \quad (2.77)$$

Now from the (anti)commutation relations Eqs.(2.68-2.70) one can recover the correct Heisenberg relations for all dynamical quantum fields $\varphi_J = (A_k, A_-, \Pi, \pi, a_-)$

$$\partial_i \varphi_J = -i [P^i, \varphi_J] \quad (2.78)$$

$$\partial_- \varphi_J = i [P^+, \varphi_J] \quad (2.79)$$

and this confirms the translation invariance of QED in the Weyl gauge. We note that the generators P^+ and P^i are not invariant under the residual gauge transformation with gauge function $h(\vec{x})$

$$P_h^+ = \Omega_h P^+ \Omega_h^\dagger = P^+ + \int d^3x G(\vec{x}) \partial_- h(\vec{x}) \quad (2.80)$$

$$P_h^i = \Omega_h P^i \Omega_h^\dagger = P^i - \int d^3x G(\vec{x}) \partial_i h(\vec{x}) \quad (2.81)$$

and this is connected with the lack of gauge invariance of the canonical energy-momentum tensor. We return to this below.

3 Implementing Gauß' law

In the previous section we derived the canonical formulation of light-front QED. The quantum Hamiltonian Eq.(2.73) is supplemented with Gauß law operator Eq.(2.74), which is to be implemented as a constraint on the physical states,

$$G(\vec{x})|\text{phys}\rangle = 0. \quad (3.1)$$

Being the generator of residual gauge transformations (see Appendix D) the Gauß law operator commutes with the Hamiltonian and, consequently, time evolution leaves the physical space invariant. The presence of Eq.(3.1) means that there are still redundant variables in the theory, which, in principle, can be eliminated in the physical sector of Hilbert space. We now apply ‘quantum mechanical gauge fixing’ [12] to the light-front formulation in order to arrive at the ‘light-cone gauge representation’ (to be defined below) of the physical Hamiltonian.

The general principle of this method is to construct a unitary gauge fixing transformation, which acts as a gauge transformation on the degrees of freedom to be kept in the theory. The gauge function, however, is a functional of the field variable to be eliminated from the Hamiltonian. The appropriate choice of the functional indeed will make the transformed Hamiltonian independent of that variable, i.e. it becomes ‘cyclic’. Concomitantly, due to the underlying gauge invariance, the transformed Gauß law operator enables one to eliminate the corresponding conjugate momentum in the physical space. It turns out that the most convenient variable to treat in this manner is the V_- field. For this reason, the final result will be said to be in ‘light-cone gauge representation’.

First unitary gauge fixing transformation. We first choose the normal mode A_- as the variable to be eliminated and so define the following unitary gauge fixing transformation

$$U_1[\vartheta] = \exp \left(-i \int d^3x g(\vec{x}) \vartheta[\vec{x}; A_-] \right), \quad (3.2)$$

with

$$g(\vec{x}) \equiv G(\vec{x}) - \partial_- \Pi(\vec{x}) = 2\partial_- \partial_i A_i - \Delta_\perp A_- - \Delta_\perp a_- - e\sqrt{2}\psi_+^\dagger \psi_+ \quad (3.3)$$

and

$$\vartheta[\vec{x}; A_-] = i \int d^3y \mathcal{G}_{(-)}[\vec{x}, \vec{y}; 0] A_-(\vec{y}). \quad (3.4)$$

Obviously, U acts as a gauge transformation on the transverse gauge fields,

$$U_1 A_j U_1^\dagger = A_j + \partial_j \vartheta, \quad (3.5)$$

zero mode gauge fields,

$$U_1 a_- U_1^\dagger = a_-, \quad (3.6)$$

$$U_1 \pi U_1^\dagger = \pi - \Delta_\perp \int_{-L}^L dy^- \vartheta[y^-, x_\perp; A_-] = \pi, \quad (3.7)$$

and the fermions

$$U_1 \psi_+ U_1^\dagger = \exp(-ie\vartheta) \psi_+. \quad (3.8)$$

It leaves A_- invariant. Due to the A_- dependence of the gauge function the transformation of Π is non-trivial

$$U_1 \Pi U_1^\dagger = \Pi - i(\mathcal{G}_{(-)}[0] * g)(\vec{x}) - 2\Delta_\perp \vartheta. \quad (3.9)$$

The coefficient of $\Delta_\perp \vartheta$ in the last term stems from the non-commutativity of $g(\vec{x})$ and $g(\vec{y})$. The transformation of the operators appearing in the fermionic part of the Hamiltonian effectively eliminates A_- from them

$$U_1 \int d^3x \xi^\dagger(\vec{x}) (\mathcal{G}_{(-)}[V_-] * \xi)(\vec{x}) U_1^\dagger = \int d^3x \xi^\dagger(\vec{x}) (\mathcal{G}_{(-)}[a_- + q_-] * \xi)(\vec{x}). \quad (3.10)$$

The transformed Gauß law operator reads

$$U_1 G U_1^\dagger = \partial_- \Pi + \frac{1}{2L} \int dx^- g(\vec{x}). \quad (3.11)$$

By integrating over x^- , one can project out two separate constraints on the transformed physical states

$$\Pi|\text{phys}'\rangle = 0 \quad (3.12)$$

and

$$\int \frac{dx^-}{2L} g(\vec{x}) |\text{phys}'\rangle = 0, \quad (3.13)$$

where

$$|\text{phys}'\rangle \equiv U_1 |\text{phys}\rangle. \quad (3.14)$$

The first constraint Eq.(3.12) can readily be implemented at this point. Now we have all the ingredients to calculate the transformed Hamilton operator acting in the physical sector of Hilbert space. The result, after the first unitary gauge fixing transformation, is

$$U_1 H U_1^\dagger |\text{phys}'\rangle = H' |\text{phys}'\rangle, \quad (3.15)$$

with

$$\begin{aligned} H' &= \int d^3x \left[\frac{1}{2} \left(\partial_j A_j - i\sqrt{2}e\mathcal{G}_-[0] * (\psi_+^\dagger \psi_+) \right)^2 + \frac{1}{4} (\partial_i A_j - \partial_j A_i)^2 \right] \\ &+ \frac{1}{16LL_\perp^2} (p^-)^2 + \frac{1}{\sqrt{2}} \int d^3x \xi^\dagger(\vec{x}) (\mathcal{G}_{(-)}[a_- + q_-] * \xi)(\vec{x}) \\ &+ \frac{1}{2} \frac{e^2}{2L} \int d^2x_\perp \Gamma^i(x_\perp) (\mathcal{G}_{(\perp)}[\mathcal{M}'^2] * \Gamma^i)(x_\perp). \end{aligned} \quad (3.16)$$

The operators Γ' and \mathcal{M}' can be obtained from the operators Γ and \mathcal{M} by replacing V_- with $a_- + q_-$ in their respective definitions, namely Eq.(2.61). Because of the fact that it

is impossible to invert completely the operator ∂_- a residual Gauß law operator makes its appearance through Eq.(3.13), namely we have the operator

$$G_2(x_\perp) = -\Delta_\perp a_-(x_\perp) - \frac{1}{2L}\rho_2(x_\perp), \quad (3.17)$$

where the two dimensional charge density is defined by

$$\rho_2 = \sqrt{2}e \int dx^- \psi_+^\dagger \psi_+. \quad (3.18)$$

G_2 commutes with the Hamiltonian and generates x^+ and x^- independent gauge transformations. Explicitly one has

$$\begin{aligned} \Omega_2[\gamma] &= \exp\left(i \int d^2x_\perp 2LG_2(x_\perp)\gamma(x_\perp)\right) \\ &= \exp\left(-i \int d^3x \left(\Delta_\perp a_-(x_\perp) + \sqrt{2}e\psi_+^\dagger(\vec{x})\psi_+(\vec{x})\right) \gamma(x_\perp)\right), \end{aligned} \quad (3.19)$$

which yields

$$\Omega_2\psi_+(\vec{x})\Omega_2^\dagger = \exp(i e \gamma(x_\perp))\psi_+(\vec{x}), \quad (3.20)$$

and

$$\Omega_2\pi(x_\perp)\Omega_2^\dagger = \pi(x_\perp) + 2L\Delta_\perp\gamma(x_\perp). \quad (3.21)$$

The other operators are invariant, and the invariance of H is easily checked. Let us also consider the momentum operators (given by Eqs.(2.76-2.77)). Straightforward but rather lengthy calculations yield in the physical sector of Hilbert space

$$U_1 P^i U_1^\dagger = \int d^3x \left(-\partial_- A_j \partial_i A_j - i\sqrt{2}\psi_+^\dagger \partial_i \psi_+ - \frac{1}{2L}\pi \partial_i a_- \right), \quad (3.22)$$

and

$$U_1 P^+ U_1^\dagger = \int d^3x \left(\partial_- A_j \partial_- A_j + i\sqrt{2}\psi_+^\dagger \partial_- \psi_+ \right). \quad (3.23)$$

The latter is invariant under the residual gauge transformations given by Eq.(3.19),

$$\Omega_2 P^+ \Omega_2^\dagger = P^+, \quad (3.24)$$

but for the transverse components one has

$$\Omega_2 P^i \Omega_2^\dagger = P^i - 2L \int d^2x_\perp G_2(x_\perp) \partial_i \gamma(x_\perp). \quad (3.25)$$

It is instructive to recall the derivation in [12]: there the ‘axial gauge representation’ was given for the equal-time formulation of QED. Up to this point it is rather analogous with the present case if one identifies the ‘-’ components with the ‘3’ components of gauge and electric fields.

Second unitary gauge fixing transformation. The condition Eq.(3.13) reflects that there remain redundant variables in the theory. A second unitary gauge fixing transformation

is in order and the strategy remains in principle the same. However, now a striking difference between the light-cone and equal-time formulation appears. In the equal time framework the residual Gauß law is independent of the zero modes fields in the 3-direction, which means that these degrees of freedom cannot be eliminated, i.e. are physical. In contrast, on the light-cone the residual Gauß law still contains a_- ; in fact (apart from the fermionic charge density) it *only* contains that particular zero mode. Excluding unitary gauges involving fermions, this leaves no other choice as the complete light-cone gauge representation. This means we will eliminate a_- and its conjugate momentum π via a second unitary gauge fixing transformation and resolution of the residual Gauß law operator.

Applying the same principle as above, we define the transformation

$$U_2 = \exp \left(i \int d^2 x_\perp \rho_2(x_\perp) \eta[x_\perp; \pi] \right), \quad (3.26)$$

where η simply is

$$\eta[x_\perp; \pi] = \frac{-1}{2L} (\mathcal{G}_{(\perp)}[0] * \pi)(x_\perp). \quad (3.27)$$

It acts as a gauge transformation in the fermionic sector

$$U_2 \psi_+ U_2^\dagger = \exp(-ie\eta) \psi_+. \quad (3.28)$$

It leaves π invariant and transforms a_- as follows

$$U_2 a_-(x_\perp) U_2^\dagger = a_-(x_\perp) - \frac{1}{2L} (\mathcal{G}_{(\perp)}[0] * \rho_2)(x_\perp). \quad (3.29)$$

In the fermionic sector we obtain

$$U_2 \xi(\vec{x}) U_2^\dagger = \exp(-ie\eta[x_\perp; \pi]) \chi(\vec{x}) = \exp \left(i \frac{e}{2L} (\mathcal{G}_{(\perp)}[0] * \pi)(x_\perp) \right) \chi(\vec{x}), \quad (3.30)$$

with

$$\chi = [m\gamma_0 - i\alpha^i \partial_i + e\alpha^i A_i] \psi_+. \quad (3.31)$$

Now an additional complication arises in the transformation of the appearing composite operators. For instance, upon transforming the first ξ -dependent term in Eq.(3.16) one explicitly encounters the expression

$$\begin{aligned} U_2 \int d^3 x \xi^\dagger(\vec{x}) \mathcal{G}_{(-)}[a_- + q_-] * \xi(\vec{x}) U_2^\dagger = \\ \int d^3 x \chi^\dagger(\vec{x}) \exp(i e \eta[x_\perp; \pi]) \left(\mathcal{G}_{(-)}[a_- - \frac{1}{2L} \mathcal{G}_{(\perp)}[0] * \rho_2 + q_-] * \exp(-ie\eta[x_\perp; \pi]) \chi \right) (\vec{x}). \end{aligned} \quad (3.32)$$

Similar expressions hold for the operators Γ' and \mathcal{M}' . Since a_- and π do not commute, we cannot cancel the exponential functions in order to eliminate π without an additional contribution: the zero mode field a_- gets shifted by a singular object, i.e.

$$\exp(i e \eta[x_\perp; \pi]) a_-(x_\perp) \exp(-ie\eta[x_\perp; \pi]) = a_-(x_\perp) - \frac{e}{2L} \mathcal{G}_{(\perp)}[x_\perp, x_\perp; 0]. \quad (3.33)$$

Here we implicitly assume some regularisation of the small distance singularity involved in $\mathcal{G}_{(\perp)}[x_{\perp}, x_{\perp}; 0]$. It is, however, not necessary to be more explicit since this singular *c-number* appears everywhere in combination with the global zero mode q_{-} . Hence it is possible to absorb the singular term into a redefinition of this global zero mode:

$$q'_{-} = q_{-} - \frac{e}{2L} \mathcal{G}_{(\perp)}[x_{\perp}, x_{\perp}; 0]. \quad (3.34)$$

Obviously the new global zero mode, q'_{-} , has the same commutation relations as q_{-} . This procedure completely cures the potential problem in eliminating π from the Hamiltonian. Eq.(3.29) immediately yields for the residual Gauß law operator

$$U_2 G_2 U_2^{\dagger} = -\mathcal{G}_{(\perp)}[0] * a_{-} + \frac{\sqrt{2}e}{2L} \Delta_{\perp} \int dx^{-} \mathcal{G}_{(\perp)}[0] * (\psi_{+}^{\dagger} \psi_{+}) - \frac{1}{2L} \rho_2 \quad (3.35)$$

Again it separates into two constraints on the transformed physical states

$$a_{-}(x_{\perp})|\text{phys}''\rangle = 0, \quad (3.36)$$

and

$$Q|\text{phys}''\rangle = \int d^2 x \rho_2(x_{\perp})|\text{phys}''\rangle = 0, \quad (3.37)$$

with

$$|\text{phys}''\rangle \equiv U_2 |\text{phys}'\rangle. \quad (3.38)$$

Eq.(3.37), the neutrality condition, will be the only constraint left in this formulation (out of the infinitely many we had at the start). The other constraint, Eq.(3.36), can be readily implemented at this point. Note that the transformed Hamiltonian does not depend on the conjugate variable π anymore. In this way we finally arrive at the following Hamiltonian

$$\begin{aligned} H_{fin} &= \int d^3 x \left[\frac{1}{2} \left(\partial_j A_j - i\sqrt{2}e \mathcal{G}_{(-)}[0] * (\psi_{+}^{\dagger} \psi_{+}) \right)^2 + \frac{1}{4} (\partial_i A_j - \partial_j A_i)^2 \right] \\ &+ \frac{1}{16LL_{\perp}^2} (p^{-})^2 + \frac{1}{\sqrt{2}} \int d^3 x \chi^{\dagger}(\vec{x}) \left(\mathcal{G}_{(-)} \left[q'_{-} - \mathcal{G}_{(\perp)}[0] * \rho_2 \right] * \chi \right) (\vec{x}) \\ &+ \frac{1}{2} \frac{e^2}{2L} \int d^2 x_{\perp} \Gamma'^{ii}(x_{\perp}) \left(\mathcal{G}_{(\perp)}[\mathcal{M}''^2] * \Gamma'^{ii} \right) (x_{\perp}), \end{aligned} \quad (3.39)$$

acting in the physical sector of the Hilbert space. The operator Γ'^{ii} is defined in the same way as Γ (cf. Eq.(2.61)), however with ξ and V_{-} replaced by χ , $q'_{-} - \mathcal{G}_{(\perp)}[0] * \rho_2$ respectively. The second replacement also yields the operator \mathcal{M}'' starting from the definition of \mathcal{M} . Apart from the global constraint, Eq.(3.37), which eventually projects on the charge zero sector, all the constraints are implemented. In other words, H_{fin} is indeed formulated in terms of unconstrained degrees of freedom. Eq.(3.39) is the main result of this paper.

The Translation Generators. Finally, we return to the momentum operators. Although they were not gauge invariant in the large Hilbert space it is clear from Eqs.(2.81,3.1)

that they are invariant in the physical sector. Therefore one can already anticipate that, finally, they can be expressed in terms of physical variables only. Indeed, we obtain in the physical space

$$U_2 U_1 P^i U_1^\dagger U_2^\dagger = - \int d^3x \left(\partial_- A_j \partial_i A_j + i\sqrt{2}\psi_+^\dagger \partial_i \psi_+ \right) \equiv P_{fin}^i, \quad (3.40)$$

and

$$U_2 U_1 P^+ U_1^\dagger U_2^\dagger = \int d^3x \left(\partial_- A_j \partial_- A_j + i\sqrt{2}\psi_+^\dagger \partial_- \psi_+ \right) \equiv P_{fin}^+, \quad (3.41)$$

as the generators of translations.

4 Displacement Symmetry

Apart from the gauge transformations considered so far, the canonical Weyl gauge Hamiltonian, cf. Eq.(2.73), is invariant under displacements. These are gauge transformations *not* generated by Gauß' law. This displacement symmetry is given by the following unitary operator

$$\Omega_d = \exp \left[-i \int d^3x \left(e\sqrt{2}\psi^\dagger(\vec{x})\psi(\vec{x}) + p^- \partial_- \right) \vec{\beta} \cdot \vec{x} \right], \quad (4.1)$$

where

$$\beta_- = \frac{\pi}{eL} n, \quad n = \pm 1, \pm 2, \dots \quad \beta^i = \frac{\pi}{eL_\perp} m^i \quad m^i = \pm 1, \pm 2, \dots \quad (4.2)$$

Under displacements only the fermion fields and the zero mode gauge field variable, q_- , transform

$$\Omega_d \psi_+(\vec{x}) \Omega_d^\dagger = e^{i e \vec{\beta} \cdot \vec{x}} \psi_+(\vec{x}) \quad (4.3)$$

$$\Omega_d q_- \Omega_d^\dagger = q_- - \beta_-. \quad (4.4)$$

Compatibility of Eqs.(4.3-4.4) with the boundary conditions for fermion and boson fields, respectively, forces the form of the gauge function β^μ given in Eq.(4.2). Now the transformation properties of composite fermion operators follow immediately

$$\Omega_d \xi_+(\vec{x}) \Omega_d^\dagger = e^{i e \vec{\beta} \cdot \vec{x}} \left(\xi_+(\vec{x}) + e\alpha^i \beta^i \psi_+(\vec{x}) \right) \quad (4.5)$$

$$\Omega_d \Gamma^i(x_\perp) \Omega_d^\dagger = \Gamma^i(x_\perp) + e\beta^i \mathcal{M}^2(x_\perp). \quad (4.6)$$

We can now see how the Hamiltonian of Eq.(2.73) changes under displacements. Only its fermionic (ξ dependent) part transforms in a non-trivial way:

$$\Omega_d H_{fer}^{eff} \Omega_d^\dagger = H_{fer}^{eff} + \delta_1 H_{fer}^{eff} + \delta_2 H_{fer}^{eff}, \quad (4.7)$$

where $\delta_1 H_{fer}^{eff}$ and $\delta_2 H_{fer}^{eff}$ are linear and quadratic in $\vec{\beta}$ respectively

$$\begin{aligned}\delta_1 H_{fer}^{eff} &= \frac{e}{\sqrt{2}} \int d^3 x \xi^\dagger \alpha^i \beta^i (\mathcal{G}_{(-)}[V_-] * \psi_+) + \frac{e}{\sqrt{2}} \int d^3 x \psi_+^\dagger \beta^i \alpha^i (\mathcal{G}_{(-)}[V_-] * \xi) \\ &+ \frac{1}{2} \frac{e^3}{2L} \int d^2 x_\perp \beta^i \left[\mathcal{M}^2(\mathcal{G}_{(\perp)}[\mathcal{M}^2] * \Gamma^i) + \Gamma^i(\mathcal{G}_{(\perp)}[\mathcal{M}^2] * \mathcal{M}^2) \right] \\ &= e \beta^i \int d^2 x_\perp \left[\Gamma^i + \frac{e^2}{2L} \mathcal{M}^2(\mathcal{G}_{(\perp)}[\mathcal{M}^2] * \Gamma^i) \right]\end{aligned}\quad (4.8)$$

$$\begin{aligned}\delta_2 H_{fer}^{eff} &= \frac{e^2}{\sqrt{2}} \int d^3 x \psi_+^\dagger \alpha^j \beta^j \alpha^i \beta^i (\mathcal{G}_{(-)}[V_-] * \psi_+) \\ &+ \frac{1}{2} \frac{e^4}{2L} \int d^2 x_\perp \beta^i \beta^i \mathcal{M}^2(\mathcal{G}_{(\perp)}[\mathcal{M}^2] * \mathcal{M}^2) \\ &= \frac{e^2}{2} \beta^i \beta^i \int d^2 x_\perp \mathcal{M}^2 \left[1 + \frac{e^2}{2L} (\mathcal{G}_{(\perp)}[\mathcal{M}^2] * \mathcal{M}^2) \right].\end{aligned}\quad (4.9)$$

Using Eq.(2.62) and periodic boundary conditions, one can show that these vanish separately:

$$\begin{aligned}\delta_1 H_{fer}^{eff} &= e \beta^i \int d^2 x_\perp \left[\Gamma^i + \frac{e^2}{2L} \mathcal{M}^2(\mathcal{G}_{(\perp)}[\mathcal{M}^2] * \Gamma^i) \right] \\ &= e \beta^i \int d^2 x_\perp \Delta_\perp(\mathcal{G}_{(\perp)}[\mathcal{M}^2] * \Gamma^i) = 0\end{aligned}\quad (4.10)$$

$$\begin{aligned}\delta_2 H_{fer}^{eff} &= \frac{e^2}{2} \beta^i \beta^i \int d^2 x_\perp \left[\mathcal{M}^2 + \frac{e^2}{2L} \mathcal{M}^2(\mathcal{G}_{(\perp)}[\mathcal{M}^2] * \mathcal{M}^2) \right] \\ &= \frac{e^2}{2} \beta^i \beta^i \int d^2 x_\perp \Delta_\perp(\mathcal{G}_{(\perp)}[\mathcal{M}^2] * \mathcal{M}^2) = 0.\end{aligned}\quad (4.11)$$

In this way we have explicitly established the invariance of H under displacements.

Since the displacements are not generated by Gauß' law one expects this symmetry to be present also in the final Hamiltonian, namely after quantum mechanically gauge fixing. First, one easily proves that the displacement operator is invariant under the gauge fixing transformations

$$U_2 U_1 \Omega_d U_1^\dagger U_2^\dagger = \Omega_d. \quad (4.12)$$

Secondly, whereas the infinitely many symmetries generated by Gauß' law essentially disappear because Gauß' law is implemented, the displacement symmetry survives this process. In other words, the local gauge transformations reduce to the identity while the global displacement symmetry remains a non-trivial symmetry of H_{fin} . With the analogous transformation properties as given in Eqs.(4.8-4.11), one indeed verifies this invariance, i.e

$$\Omega_d H_{fin} \Omega_d^\dagger = H_{fin}, \quad (4.13)$$

as above. Furthermore, the momentum operators transform as follows

$$\Omega_d P_{fin}^+ \Omega_d^\dagger = P_{fin}^+ - \beta_- Q, \quad (4.14)$$

$$\Omega_d P_{fin}^i \Omega_d^\dagger = P_{fin}^i + \beta^i Q. \quad (4.15)$$

Obviously, their shift is proportional to the total charge. Therefore, the remaining global constraint, Eq.(3.37), guarantees the compatibility of translation invariance and these global residual gauge transformations in the physical space.

5 Discussion and Conclusions

First let us summarise what we have achieved in the present work: working in the light-front formalism for quantum electrodynamics, we implemented the Weyl gauge $A^- = 0$ and quantised the theory. By putting the system in a finite ‘volume’ the (possibly singular) infrared behaviour was *a priori* regularised and all the degrees of freedom could be cleanly disentangled into normal, proper zero and global zero mode sectors. By performing unitary transformations we succeeded in implementing Gauss law and deriving the light-front Hamiltonian which acts in the physical Hilbert space – albeit in a somewhat abstract form. Again, we say abstract because we have not explicitly described the nonperturbative meaning of the functional Green’s function $\mathcal{G}_{(\perp)}$. We have simply assumed existence and uniqueness of the Green’s function. Nevertheless, the final result for the Hamiltonian is actually new, and has not been given, even at this ‘formal’ level, including the global zero modes. It is the inclusion of these fields that allowed us to verify the invariance of the theory under global (large gauge transformation) displacements. We can thus confirm that the symmetry works just as well on the light-front as it does in standard quantisation.

We now compare the significance of the displacement symmetry in the present light-front formulation with what was learned in [12] for equal-time. On the light-cone only the ‘ $-$ ’ component of the photon field is affected by the displacements. However, in the instant-form all three components of the gauge field are shifted. As was pointed out in [12] the displacement symmetry actually can be understood from Maxwell’s equations by identifying the zero-mode of the displacement vector as conserved quantity. In light-front coordinates the relevant Maxwell equations read in full

$$\partial_+ (\partial_+ V_- - \partial_i V_i) = \mathcal{J}^- , \quad (5.1)$$

$$(2\partial_+ \partial_- - \Delta_\perp) V_i = \partial_i (\partial_+ V_- - \partial_j V_j) + \mathcal{J}^i . \quad (5.2)$$

By means of the continuity equation we rewrite these as

$$\partial_+ (\partial_+ V_- - \partial_i V_i) = \partial_- (x^- \mathcal{J}^-) + \partial_i (x^- \mathcal{J}^i) + x^- \partial_+ \mathcal{J}^+ , \quad (5.3)$$

$$(2\partial_+ \partial_- - \Delta_\perp) V_i = \partial_i (\partial_+ V_- - \partial_i V_i) + \partial_- (x^i \mathcal{J}^-) + \partial_j (x^i \mathcal{J}^j) + x^i \partial_+ \mathcal{J}^+ . \quad (5.4)$$

Integration over space and dropping surface terms such as $x^- \mathcal{J}^-|_{-\infty}^\infty$, namely assuming

the current to be localised, leads to the two equations

$$\partial_+^2 q_- = \int d^3x x^- \partial_+ \mathcal{J}^+ , \quad (5.5)$$

$$0 = \int d^3x x^i \partial_+ \mathcal{J}^+ . \quad (5.6)$$

(Of course, one can be more precise concerning the omission of surface terms and impose periodic boundary conditions [12]. This yields the conservation law in exponentiated form, corresponding to Eq.(4.1), but for our present purpose this difference is not relevant.) In this way we obtain the following conserved quantities

$$\partial_+ \left(p^- - \int d^3x x^- \mathcal{J}^+ \right) = 0 , \quad (5.7)$$

$$\partial_+ \int d^3x x^i \mathcal{J}^+ = 0 . \quad (5.8)$$

Indeed only the light-cone component of the electric field, namely p^- , appears in the conserved quantities since the transverse zero mode components are not dynamical. In particular, this means that for free Maxwell theory there is only *one* non-trivial displacement symmetry.

The latter feature obscures an interpretation analogous to the one in the equal-time theory. There it was argued that, for zero or weak coupling, the displacement symmetry is spontaneously broken, giving rise to zero-mass particles: the physical photons. The argument for free Maxwell theory was based on the formal analogy with free massless scalar theory, where the massless scalars can indeed be interpreted as Goldstone bosons [19]. Actually, even for free massless scalar light-cone theory the interpretation of the masslessness in terms of the Goldstone mechanism is unclear. This may indicate either a possible pathology in massless non-interacting theories on the light-cone or the need for further studies of the Goldstone mechanism and its consequences in that framework. We note one work in this direction by [20].

Of course, in the interacting light-cone theory the situation may actually be similar to equal-time QED. In other words, the displacement symmetries may be spontaneously broken leading to photons as Goldstone particles. This is supported by the empirical fact that photons are massless. However, to see this in, say, perturbation theory is difficult given the above problems already at *zeroth* order. Up to now, other theoretical arguments are also lacking. Nevertheless, the formal developments presented here, can serve as the starting point for such interesting investigations.

Acknowledgements

One of us (JP) thanks MPI for hospitality and DFG-PAC Exchange Programme for financial support while visiting Heidelberg where part of this work was completed. (ACK) and

(HVLN) were supported by a *Max-Planck Gesellschaft* Stipendium. (ACK) also thanks the Department of Physics of the University of Tasmania for its hospitality while this work was completed.

Appendix A. Construction of Green's Function $\mathcal{G}_{(-)}$.

To construct the Green's function $\mathcal{G}_{(-)}$ we follow the procedure outlined in [13] and give the eigenfunctions to the operator iD_- :

$$iD_- \zeta_n(\vec{x}) = \lambda_n(x_\perp) \zeta_n(\vec{x}), \quad (\text{A.1})$$

$$\zeta_n(\vec{x}) = \exp \left[-ie \left(\tau(\vec{x}) - x^- v(x_\perp) + \frac{2\pi n x^-}{2Le} + \Delta(x_\perp) \right) \right], \quad (\text{A.2})$$

$$\lambda_n(x_\perp) = \frac{2\pi n}{2L} - ev(x_\perp) \quad (\text{A.3})$$

$$\tau(\vec{x}) = \int_{-L}^{x^-} dz^- V_-(z^-, x_\perp), \quad (\text{A.4})$$

$$v(x_\perp) = \frac{1}{2L} \int_{-L}^L dx^- V_-(\vec{x}), \quad (\text{A.5})$$

with Δ an arbitrary local phase and n any integer. A useful relation to show that ζ_n and λ_n satisfy Eq.(A.1) is

$$i\partial_- \zeta_n(\vec{x}) = e \left[V(\vec{x}) - v(x_\perp) + \frac{2\pi n}{2eL} \right] \zeta_n(\vec{x}) \quad (\text{A.6})$$

which itself follows easily from Eq.(A.2). We see that both the eigenfunctions ζ_n and eigenvalues λ_n are operator valued. The theory is Abelian and $[V_-(\vec{x}), V_-(\vec{y})] = 0$ so there are no ordering ambiguities. The phase Δ can be chosen such that the V_- dependence in ζ_n is given by just the periodic step function $\tilde{\vartheta}$, used in [12], namely

$$\Delta(x_\perp) = \int_{-L}^L dy^- \frac{y^- - L}{2L} V_-(y^-, x_\perp). \quad (\text{A.7})$$

The eigenfunctions are, by construction, periodic. As one can explicitly verify, this set of eigenfunctions is orthonormal and complete

$$\frac{1}{2L} \int_{-L}^L dx^- \zeta_n^*(\vec{x}) \zeta_m(\vec{x}) = \delta_{nm}, \quad (\text{A.8})$$

$$\frac{1}{2L} \sum_n \zeta_n(x^-, x_\perp) \zeta_n^*(y^-, x_\perp) = \delta(x^- - y^-). \quad (\text{A.9})$$

Let us now consider the differential equation

$$iD_- F = K \quad (\text{A.10})$$

where F and K are periodic functions. They can thus be expanded in the eigenfunctions ζ_n . Then we readily obtain a relation for the respective expansion coefficients $f_n(x_\perp), k_n(x_\perp)$ of F and K :

$$\lambda_n f_n = k_n . \quad (\text{A.11})$$

For non-zero eigenvalues this equation is trivially solved. If eigenvalue zero appears in the spectrum, $\lambda_{n_0} = 0$, we conclude the following: First, the zero mode (with respect to the covariant derivative) of K , i.e. k_{n_0} , must vanish in order that the differential equation indeed has solutions. Secondly, even in that case the zero mode of F , f_{n_0} , cannot be solved for from Eq.(A.11). Assuming that, either there is no eigenvalue zero, or that if $\lambda_{n_0} = 0$ then $k_{n_0} = 0$, leads to the explicit solution

$$F(\vec{x}) = \sum_{n \neq n_0} \frac{\zeta_n(\vec{x})}{2L\lambda_n} \int_{-L}^L dy^- \zeta_n^\dagger(y^-, x_\perp) K(y^-, x_\perp). \quad (\text{A.12})$$

Thus we can identify a Green's function

$$\mathcal{G}(x^-, y^-; x_\perp) = \sum_{n \neq n_0} \frac{\zeta_n(\vec{x}) \zeta_n^\dagger(y^-, x_\perp)}{2L\lambda_n}. \quad (\text{A.13})$$

The Green's function defined in the main text follows trivially

$$\mathcal{G}_{(-)}[\vec{x}, \vec{y}; V_-] = \delta^{(2)}(x_\perp - y_\perp) \mathcal{G}(x^-, y^-; x_\perp). \quad (\text{A.14})$$

It should be emphasized that the basis functions ζ_n as well as the eigenvalues λ_n depend on the dynamical variable V_- . The Green's function satisfies

$$iD_- \mathcal{G}(x^-, y^-) = \delta(x^- - y^-) - \frac{1}{2L} \sum_{n_0} \zeta_{n_0}(\vec{x}) \zeta_{n_0}^\dagger(y^-, x_\perp). \quad (\text{A.15})$$

Recall that $\lambda_{n_0} = 0$. The simplest example is, of course, $V_- = 0$. Then $n_0 = 0$ and the corresponding eigenfunction is $\zeta_0 = 1$. Thus (including the perpendicular delta function)

$$i\partial_- \mathcal{G}_{(-)}[\vec{x}, \vec{y}; 0] = \delta^{(2)}(x^\perp - y^\perp) \left[\delta(x^- - y^-) - \frac{1}{2L} \right], \quad (\text{A.16})$$

and

$$\mathcal{G}_{(-)}[\vec{x}, \vec{y}; 0] = -i\delta^2(x_\perp - y_\perp) \sum_{n \neq 0} \frac{1}{2i\pi n} \exp\left(\frac{2i\pi n(x^- - y^-)}{2L}\right), \quad (\text{A.17})$$

where one readily recognises the periodic step function $\tilde{\vartheta}$ in the sum on the right hand side. Analogous expressions can be found for antiperiodic functions by replacing $2n \rightarrow (2n + 1)$ in Eqs.(A.1,A.2,A.3). For the example above, $V_- = 0$, one obtains

$$i\partial_- \mathcal{G}_{(-)}^a[\vec{x}, \vec{y}; 0] = \delta^{(2)}(x_\perp - y_\perp) \delta^a(x^- - y^-), \quad (\text{A.18})$$

and

$$\mathcal{G}_{(-)}^a[\vec{x}, \vec{y}; 0] = -i\delta^2(x_\perp - y_\perp) \sum \frac{1}{(2n+1)i\pi} \exp\left(\frac{(2n+1)i\pi(x^- - y^-)}{2L}\right) \quad (\text{A.19})$$

for antiperiodic boundary conditions.

With the explicit form of the Green's function it is straightforward to study the effect of the unitary gauge fixing transformations on it, thereby verifying the results given in the main text. Consider, for example, the first unitary transformation. As it is, $\mathcal{G}_{(-)}$ is obviously invariant under the U_1 of Eq.(3.2). However, since it is 'sandwiched' between the fermion operators ξ , it picks up the (dynamical) phases $e\vartheta$. One obtains

$$\begin{aligned} \mathcal{G}_{(-)}[\vec{x}, \vec{y}; V_-] &\rightarrow \exp(i e \vartheta(\vec{x})) \mathcal{G}_{(-)}[\vec{x}, \vec{y}; V_-] \exp(-i e \vartheta(\vec{y})) \\ &= \exp\left(-e \int d^3 z \mathcal{G}_{(-)}[\vec{x}, \vec{z}; 0] A_-(\vec{z})\right) \mathcal{G}_{(-)}[\vec{x}, \vec{y}; V_-] \exp\left(-e \int d^3 z \mathcal{G}_{(-)}[\vec{x}, \vec{z}; 0] A_-(\vec{z})\right) \\ &= \exp\left(i e \int dz^- \tilde{\vartheta}(x^- - z^-) A_-(x_\perp, z^-)\right) \mathcal{G}_{(-)}[\vec{x}, \vec{y}; V_-] \times \\ &\quad \exp\left(-i e \int dz^- \tilde{\vartheta}(y^- - z^-) A_-(y_\perp, z^-)\right) \\ &= \mathcal{G}_{(-)}[\vec{x}, \vec{y}; a_- + q_-] \end{aligned} \quad (\text{A.20})$$

in explicit detail.

Appendix B. Decomposition of electromagnetic currents.

The electromagnetic current

$$\mathcal{J}^\mu = -e \bar{\psi} \gamma^\mu \psi \quad (\text{B.1})$$

satisfies periodic boundary conditions and therefore can be decomposed into its global zero modes

$$Q^+ = -e\sqrt{2} \int \frac{d^3 x}{8L L_\perp^2} \psi_+^\dagger(\vec{x}) \psi_+(\vec{x}) \quad (\text{B.2})$$

$$Q^- = -e\sqrt{2} \int \frac{d^3 x}{8L L_\perp^2} \psi_-^\dagger(\vec{x}) \psi_-(\vec{x}) \quad (\text{B.3})$$

$$Q^i = -e \int \frac{d^3 x}{8L L_\perp^2} \left[\psi_+^\dagger(\vec{x}) \alpha^i \psi_-(\vec{x}) + \psi_-^\dagger(\vec{x}) \alpha^i \psi_+(\vec{x}) \right], \quad (\text{B.4})$$

proper zero modes

$$j^+(x_\perp) = -e\sqrt{2} \int_{-L}^L \frac{dx^-}{2L} \psi_+^\dagger(\vec{x}) \psi_+(\vec{x}) - Q^+ \quad (\text{B.5})$$

$$j^-(x_\perp) = -e\sqrt{2} \int_{-L}^L \frac{dx^-}{2L} \psi_-^\dagger(\vec{x}) \psi_-(\vec{x}) - Q^- \quad (\text{B.6})$$

$$j^i(x_\perp) = -e \int_{-L}^L \frac{dx^-}{2L} \left[\psi_+^\dagger(\vec{x}) \alpha^i \psi_-(\vec{x}) + \psi_-^\dagger(\vec{x}) \alpha^i \psi_+(\vec{x}) \right] - Q^i \quad (\text{B.7})$$

and the normal modes

$$J^+(\vec{x}) = -e\sqrt{2}\psi_+^\dagger(\vec{x})\psi_+(\vec{x}) - j^+(x_\perp) - Q^+ \quad (\text{B.8})$$

$$J^-(\vec{x}) = -e\sqrt{2}\psi_-^\dagger(\vec{x})\psi_-(\vec{x}) - j^-(x_\perp) - Q^- \quad (\text{B.9})$$

$$J^i(\vec{x}) = -e \left[\psi_+^\dagger(\vec{x})\alpha^i\psi_-(\vec{x}) + \psi_-^\dagger(\vec{x})\alpha^i\psi_+(\vec{x}) \right] - j^i(x_\perp) - Q^i. \quad (\text{B.10})$$

Through the constraint for ψ_- these will involve the photon degrees of freedom.

Appendix C. Momenta, Constraints and Dirac Brackets.

Using the fermionic Lagrangian Eq.(2.54) one has the canonical momenta

$$\Pi_{\psi_+} = \frac{\delta L_{fer}}{\delta(\partial_+\psi_+)} = -i\sqrt{2}\psi_+^\dagger \quad (\text{C.1})$$

$$\Pi_{\psi_-} = \frac{\delta L_{fer}}{\delta(\partial_+\psi_-)} = 0 \quad (\text{C.2})$$

$$\Pi_{\psi_-^\dagger} = \frac{\delta L_{fer}}{\delta(\partial_+\psi_-^\dagger)} = 0 \quad (\text{C.3})$$

$$p^i = \frac{\delta L_{fer}}{\delta(\partial_+q_i)} = 0. \quad (\text{C.4})$$

The extended canonical Hamiltonian density defined as

$$\mathcal{H}_E^{fer} = (\partial_+\psi_+)\Pi_{\psi_+} - \mathcal{L}_{fer} + \Pi_{\psi_-}v + v^\dagger\Pi_{\psi_-^\dagger} + u_i p^i, \quad (\text{C.5})$$

after some simple manipulations takes the form

$$\begin{aligned} \mathcal{H}_E^{fer} &= -i\sqrt{2}\psi_-^\dagger\partial_-\psi_- - i\psi_-^\dagger\alpha^i\partial_i\psi_+ - i\psi_+^\dagger\alpha^i\partial_i\psi_- \\ &+ m\psi_+^\dagger\gamma^0\psi_- + m\psi_-^\dagger\gamma^0\psi_+ + e\sqrt{2}\psi_-^\dagger\psi_-V_- \\ &+ eV_i\left(\psi_+^\dagger\alpha^i\psi_- + \psi_-^\dagger\alpha^i\psi_+\right) + \frac{1}{2}j^i\mathcal{G}_{(\perp)}[0] * j^i \\ &+ \Pi_{\psi_-}v + v^\dagger\Pi_{\psi_-^\dagger}u_i p^i. \end{aligned} \quad (\text{C.6})$$

In this system, there are primary constraints

$$\Phi_1 = \Pi_{\psi_-} \quad (\text{C.7})$$

$$\Phi_2 = \Pi_{\psi_-^\dagger} \quad (\text{C.8})$$

$$\Phi_3^i = p^i \quad (\text{C.9})$$

and the secondary constraints, which are given by the stationarity conditions

$$\begin{aligned} \chi_1 &= \partial_+\Phi_1 = i\sqrt{2}\partial_-\psi_-^\dagger + i\partial_i\psi_+^\dagger\alpha^i + e\sqrt{2}\psi_-^\dagger V_- \\ &+ e\psi_+^\dagger\alpha^i\left(V_i - \mathcal{G}_{(\perp)} * j^i\right) + m\psi_+^\dagger\gamma^0 \simeq 0 \end{aligned} \quad (\text{C.10})$$

$$\begin{aligned}\chi_2 &= \partial_+ \Phi_2 = i\sqrt{2}\partial_- \psi_- + i\alpha^i \partial_i \psi_+ - e\sqrt{2}\psi_- V_- \\ &\quad - e\alpha^i \left(V_i - \mathcal{G}_{(\perp)} * j^i \right) \psi_+ - m\gamma^0 \psi_+ \simeq 0\end{aligned}\tag{C.11}$$

$$\chi_3^i = \partial_+ \Phi_3^i = Q^i = -e \int d^3x \left(\psi_-^\dagger \alpha^i \psi_+ + \psi_+^\dagger \alpha^i \psi_- \right) (\vec{x}).\tag{C.12}$$

They all form a set of second-class constraints and Dirac brackets can be given for non-constrained variables

$$\left\{ \psi_+^\dagger(\vec{x}), \psi_+^\dagger(\vec{y}) \right\}_D^+ = -\frac{i}{\sqrt{2}} \Lambda^{(+)} \delta_a^{(3)}(\vec{x} - \vec{y})\tag{C.13}$$

while all others vanish.

For the total effective lagrangian Eq.(2.67) one finds the canonical momenta

$$\Pi^- = \frac{\delta L_{eff}}{\delta(\partial_+ A_-)} = \Pi\tag{C.14}$$

$$\Pi^i = \frac{\delta L_{eff}}{\delta(\partial_+ A_i)} = \partial_- A_i\tag{C.15}$$

$$\Pi_{\psi_{(+)}} = \frac{\delta L_{eff}}{\delta(\partial_+ \psi_{(+)})} = -i\sqrt{2}\psi_+^\dagger\tag{C.16}$$

$$\pi^-(x_\perp) = \frac{\delta L_{eff}}{\delta(\partial_+ a_-(x_\perp))} = \pi(x_\perp).\tag{C.17}$$

$$p^- = \frac{\delta L_{eff}}{\delta(\partial_+ q_-)} = 8LL_\perp^2 \partial_+ q_-.\tag{C.18}$$

There is only one primary constraint here, but it is second-class and renders only one Dirac bracket (C.22) different from the corresponding Poisson bracket

$$\{\Pi^-(\vec{x}), A_-(\vec{y})\}_D = -\delta_n^{(3)}(\vec{x} - \vec{y})\tag{C.19}$$

$$\left\{ \psi_{(+)}^\dagger(\vec{x}), \psi_{(+)}(\vec{y}) \right\}_D^+ = -\frac{i}{\sqrt{2}} \Lambda^{(+)} \delta_a^{(3)}(\vec{x} - \vec{y})\tag{C.20}$$

$$\{\pi(x_\perp), a_-(y_\perp)\}_D = -\delta_p^{(2)}(x_\perp - y_\perp)\tag{C.21}$$

$$\{A_i(\vec{x}), A_j(\vec{y})\}_D = -\frac{i}{2} \delta_{ij} \mathcal{G}_{(-)}[\vec{x}, \vec{y}; 0]\tag{C.22}$$

$$\{p^-, q_-\}_D = -1.\tag{C.23}$$

With these, the final quantum commutators can be formulated, as given in the text.

Appendix D. Gauß Law Operator, Residual Gauge Freedom

In the Weyl gauge, the Gauß law operator $G(\vec{x})$

$$G(\vec{x}) = \partial_- \Pi(\vec{x}) + 2\partial_- \partial_i A_i(\vec{x}) - \Delta_\perp A_-(\vec{x}) - \Delta_\perp a_-(x_\perp) - e\sqrt{2}\psi_+^\dagger(\vec{x})\psi_+(\vec{x})\tag{D.1}$$

is x^+ -independent by the Hamilton equations of motion, i.e. it commutes with the Hamiltonian. The independent fields have the following commutators with $G(\vec{x})$

$$[G(\vec{x}), A_-(\vec{y})] = -i\partial_-^x \delta_n^{(3)}(\vec{x} - \vec{y}) \quad (\text{D.2})$$

$$[G(\vec{x}), A_i(\vec{y})] = -i\partial_i^x \delta_n^{(3)}(\vec{x} - \vec{y}) \quad (\text{D.3})$$

$$[G(\vec{x}), \Pi(\vec{y})] = -i\Delta_\perp \delta_n^{(3)}(\vec{x} - \vec{y}) \quad (\text{D.4})$$

$$[G(\vec{x}), \psi_{(+)}(\vec{y})] = e\psi_+(\vec{x})\delta_a^{(3)}(\vec{x} - \vec{y}) \quad (\text{D.5})$$

$$[G(\vec{x}), a_-(y_\perp)] = 0 \quad (\text{D.6})$$

$$[G(\vec{x}), \pi(y_\perp)] = -i\Delta_\perp \delta_p^{(2)}(x_\perp - y_\perp) \quad (\text{D.7})$$

$$[G(\vec{x}), a_-(y_\perp)] = 0 \quad (\text{D.8})$$

$$[G(\vec{x}), p^-] = [G(\vec{x}), q_-] = 0. \quad (\text{D.9})$$

The Gauss law operator is the generator of the residual x^+ -independent gauge transformations with periodic gauge function $h(\vec{x})$:

$$A_i(\vec{x})^h = \Omega_h A_i(\vec{x}) \Omega_h^\dagger = A_i(\vec{x}) - \partial_i h(\vec{x}) \quad (\text{D.10})$$

$$A_-(\vec{x})^h = \Omega_h A_-(\vec{x}) \Omega_h^\dagger = A_-(\vec{x}) - \partial_- h(\vec{x}) \quad (\text{D.11})$$

$$\Pi(\vec{x})^h = \Omega_h \Pi(\vec{x}) \Omega_h^\dagger = \Pi(\vec{x}) + \Delta_\perp h(\vec{x}) \quad (\text{D.12})$$

$$\psi_+(\vec{x})^h = \Omega_h \psi_+(\vec{x}) \Omega_h^\dagger = e^{ieh(\vec{x})} \psi_+(\vec{x}) \quad (\text{D.13})$$

$$a_-(x_\perp)^h = \Omega_h a_-(x_\perp) \Omega_h^\dagger = a_-(x_\perp) \quad (\text{D.14})$$

$$\pi(x_\perp)^h = \Omega_h \pi(x_\perp) \Omega_h^\dagger = \pi(x_\perp) + \Delta_\perp \int_{-L}^L dy^- h(y^-, x_\perp) \quad (\text{D.15})$$

$$p^{-h} = \Omega_h p^- \Omega_h^\dagger = p^- \quad (\text{D.16})$$

where

$$\Omega_h = e^{i \int d^3 \vec{x} h(\vec{x}) G(\vec{x})}. \quad (\text{D.17})$$

In the above transformations for fermion fields the following identity has been used

$$\psi_+(x^-) \delta_a(x^- - y^-) = \psi_+(y^-) \left[\delta(x^- - y^-) + \frac{1}{2L} \right] \quad (\text{D.18})$$

where the delta distribution on the right hand side is now the complete expression for periodic functions. One can prove this by integrating its both sides either with a smooth test function periodic in the x^- variable or a smooth test function antiperiodic in the y^- variable.

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